COSET CONSTRUCTION FOR WINDING SUBALGEBRAS AND APPLICATIONS

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ABSTRACT

In this paper we review the coset construction for winding subalgebras of affine Lie algebras. We classify all cosets of central charge $\widehat{c} < 1$ and calculate their branching rules. The corresponding character identities give certain 'doubling formulae' for the affine characters. We discuss some applications of our construction, in particular we find a simple proof of a crucial identity needed for the computation of the level-2 root multiplicities of the hyperbolic Kac-Moody algebra E_{10}.

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Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra. Consider the (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}_k$ defined by $\widehat{\mathfrak{g}}_k = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}k \oplus \mathbb{C}d$ with commutators

$$\begin{aligned} [x(m), y(n)] &= [x,y](m+n) + k\, m\, (x|y)\, \delta_{m,-n}\,, \\ [d, x(n)] &= -n\, x(n)\,, \\ [k, x(n)] &= [k, d] \;=\; 0\,, \end{aligned} \tag{1}$$

where we have written $x(n) = x \otimes t^n$. We follow the conventions of [1]. In particular $(\ |\ )$ denotes the Killing form on both $\mathfrak{g}$ and $\mathfrak{h}^*$, normalized such that $(\theta|\theta) = 2$ for a long root θ of $\mathfrak{g}$. The integrable highest weight modules $L(\Lambda)$ of $\widehat{\mathfrak{g}}$ at level k are parametrized by dominant integral weights Λ such that $\sum a_i^\vee (\Lambda, \alpha_i^\vee) = k$.

The proof of the following theorem is standard

Theorem 1.

i. *For every $j \in \mathbb{N}$ we have an embedding $\widehat{\mathfrak{g}}_{jk} \hookrightarrow \widehat{\mathfrak{g}}_k$ defined by $x(n) \mapsto x(jn)$.*

ii. *Let $L(\Lambda)$ be an integrable highest weight module of $\widehat{\mathfrak{g}}_k$ with highest weight vector v_Λ. Considered as a $\widehat{\mathfrak{g}}_{jk}$ module, $L(\Lambda)$ is integrable (and hence fully reducible). Moreover, $U(\widehat{\mathfrak{g}}_{jk}) \cdot v_\Lambda \cap L(\Lambda)$ is an irreducible integrable $\widehat{\mathfrak{g}}_{jk}$ module.*

(Note: $\widehat{\mathfrak{g}}_{jk}$ is called a winding subalgebra of $\widehat{\mathfrak{g}}_k$ [2].)

As an example, consider $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2$. We have, under $\widehat{\mathfrak{g}}_k \hookrightarrow \widehat{\mathfrak{g}}_1$ $(k \geq 2)$, e.g.,

$$U(\widehat{\mathfrak{g}}_k) \cdot v_{\Lambda_0} \cap L(\Lambda_0) \;\cong\; L(k\Lambda_0)\,, \tag{2}$$

and

$$U(\widehat{\mathfrak{g}}_k) \cdot v_{\Lambda_1} \cap L(\Lambda_1) \;\cong\; L((k-1)\Lambda_0 + \Lambda_1)\,. \tag{3}$$

Other irreducible modules can be projected out by considering other maximal vectors (vectors in $L(\Lambda)$ on the Weyl orbit of the highest weight vector), e.g.

$$U(\widehat{\mathfrak{g}}_k) \cdot v_{r_0\Lambda_0} \cap L(\Lambda_0) \;\cong\; L((k-2)\Lambda_0 + 2\Lambda_1)\,. \tag{4}$$

We can actually do more than projecting out the irreducible modules. The $\widehat{\mathfrak{g}}_k$ modules decompose with respect to a direct sum of $\widehat{\mathfrak{g}}_{jk}$ and a coset Virasoro algebra. The construction is a slight modification of the standard coset construction [3,4].

Recall that the Virasoro algebra, V_c, is generated by $\{L(m) \,|\, m \in \mathbb{Z}\}$ and a central charge c with relations

$$[L(m), L(n)] \;=\; (m-n)\, L(m+n) + \tfrac{c}{24}\, m(m^2-1)\, \delta_{m,-n}\,, \tag{5}$$

Any (positive energy) module of $\widehat{\mathfrak{g}}_k$ can be extended to a module of the semi-direct sum $V_c \oplus \widehat{\mathfrak{g}}_k$ by means of the affine Sugawara construction [3]

$$L^{\mathfrak{g},k}(m) \;=\; \frac{1}{2(k + \mathrm{h}^\vee)} \sum_{n\in\mathbb{Z}} : x^a(m-n) x^a(n) :\,, \tag{6}$$

where $\mathrm{h}^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$, and $\{x^a,\, a = 1, \ldots, \dim \mathfrak{g}\}$ is an orthonormal basis of $\mathfrak{g}$ (with respect to $(\ |\ )$). The generators $L^{\mathfrak{g},k}(m)$ satisfy a Virasoro algebra (5) with central charge

$$c^{\mathfrak{g}}(k) \;=\; \frac{k \dim \mathfrak{g}}{k + \mathrm{h}^\vee}\,, \tag{7}$$

while

$$[L^{\mathfrak{g},k}(m), x(n)] \;=\; -n\, x(m+n)\,. \tag{8}$$

We can identify $-d = L^{\mathfrak{g},k}(0) - h(\Lambda)$ where

$$h(\Lambda) \;=\; \frac{(\Lambda, \Lambda + 2\rho)}{2(k + \mathrm{h}^\vee)}\,. \tag{9}$$

The following coset construction was first discovered by Kac and Wakimoto [2]

Theorem 2. *Let $j \in \mathbb{N}$.*

i. Define

$$L^{\mathfrak{g},k,j}(m) \;=\; \tfrac{1}{j} L^{\mathfrak{g},k}(jm) + \tfrac{c^{\mathfrak{g}}(k)}{24}\left(j - \tfrac{1}{j}\right)\delta_{m,0}\,, \tag{10}$$

where $L^{\mathfrak{g},k}(m)$ denotes the Sugawara generator (6). Then $L^{\mathfrak{g},k,j}(m)$ satisfies a Virasoro algebra with central charge $c = jc^{\mathfrak{g}}(k)$.

ii. The coset generator

$$\widehat{L}^{\mathfrak{g},k,j}(m) \;=\; L^{\mathfrak{g},k,j}(m) - L^{\mathfrak{g},jk}(m)\,. \tag{11}$$

satisfies a Virasoro algebra with central charge

$$\widehat{c}^{\mathfrak{g}}(k;j) \;=\; jc^{\mathfrak{g}}(k) - c^{\mathfrak{g}}(jk)\,, \tag{12}$$

and commutes with $\widehat{\mathfrak{g}}_{jk}$.

Corollary 3. *The irreducible integrable highest weight modules $L(\Lambda)$ of $\widehat{\mathfrak{g}}_k$ decompose into a direct sum of unitary irreducible modules of $V_{\hat{c}} \oplus \widehat{\mathfrak{g}}_{jk}$ under $V_{\hat{c}} \oplus \widehat{\mathfrak{g}}_{jk} \hookrightarrow \widehat{\mathfrak{g}}_k$.*

Irreducible highest weight modules $L(h,c)$ of the Virasoro algebra are parametrized by the central charge c and the conformal dimension h, i.e. the eigenvalue of $L(0)$ on the highest weight vector. For $0 < c < 1$ these modules are unitary provided c is in the series

$$c(m) \;=\; 1 - \frac{6}{m(m+1)}\,, \tag{13}$$

and, for given m, the conformal dimension is an element of the Kac table

$$h^{(m)}(r,s) \;=\; \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}\,, \qquad 1 \le r \le m-1\,, 1 \le s \le m\,. \tag{14}$$

The character $\mathrm{ch}^V_h(q)$ of the irreducible V_c-module $L(h,c)$ is defined by

$$\mathrm{ch}^V_h(q) \;=\; \mathrm{Tr}_{\,L(h,c)}\; q^{L(0) - c/24}\,. \tag{15}$$

For $h = h^{(m)}(r,s)$ we will also write $\mathrm{ch}^V_{(r,s)}(q)$. Similarly, the character $\mathrm{ch}_{L(\Lambda)}(z;q)$ of the $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ is defined by

$$\mathrm{ch}_{L(\Lambda)}(z;q) \;=\; \mathrm{Tr}_{\,L(\Lambda)}\; z^h\, q^{L^{\mathfrak{g},k}(0) - c^{\mathfrak{g}}(k)/24}\,, \tag{16}$$

where $z^h \equiv \prod z_i^{h_i}$ and h_i runs over a basis of the Cartan subalgebra of $\mathfrak{g}$.

Theorem 4. *The following is a complete list of all simple Lie algebras $\mathfrak{g}$ such that $\widehat{c}^{\mathfrak{g}}(k;j) < 1$ together with the branching rules of all integrable highest weight modules of $\widehat{\mathfrak{g}}_k$ under $V_{\hat{c}} \oplus \widehat{\mathfrak{g}}_{jk} \hookrightarrow \widehat{\mathfrak{g}}_k$ (up to those related by automorphisms of $\mathfrak{g}$):*

- $(A_1^{(1)})_2 \hookrightarrow (A_1^{(1)})_1$, $\widehat{c} = c(3) = \frac{1}{2}$

$$\begin{aligned} q^{\frac{1}{8}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= \mathrm{ch}^V_{(1,2)}(q^2)\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) + \mathrm{ch}^V_{(2,2)}(q^2)\mathrm{ch}_{L(2\Lambda_1)}(z;q^2) \\ q^{\frac{1}{8}}\ \mathrm{ch}_{L(\Lambda_1)}(z;q) &= (\mathrm{ch}^V_{(1,1)}(q^2) + \mathrm{ch}^V_{(2,1)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_1)}(z;q^2) \end{aligned}$$

- $(A_2^{(1)})_2 \hookrightarrow (A_2^{(1)})_1$, $\widehat{c} = c(5) = \frac{4}{5}$

$$\begin{aligned} q^{\frac{1}{4}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= (\mathrm{ch}^V_{(1,2)}(q^2) + \mathrm{ch}^V_{(4,2)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(3,2)}(q^2))\mathrm{ch}_{L(\Lambda_1+\Lambda_2)}(z;q^2) \\ q^{\frac{1}{4}}\ \mathrm{ch}_{L(\Lambda_1)}(z;q) &= (\mathrm{ch}^V_{(1,2)}(q^2) + \mathrm{ch}^V_{(4,2)}(q^2))\mathrm{ch}_{L(2\Lambda_2)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(3,2)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_1)}(z;q^2) \end{aligned}$$

- $(E_8^{(1)})_2 \hookrightarrow (E_8^{(1)})_1$, $\widehat{c} = c(3) = \frac{1}{2}$

$$\begin{aligned} q\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= \mathrm{ch}^V_{(2,1)}(q^2)\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) + \mathrm{ch}^V_{(2,2)}(q^2)\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\ &+ \mathrm{ch}^V_{(2,3)}(q^2)\mathrm{ch}_{L(\Lambda_7)}(z;q^2) \end{aligned}$$

- $(E_7^{(1)})_2 \hookrightarrow (E_7^{(1)})_1$, $\widehat{c} = c(4) = \frac{7}{10}$

$$\begin{aligned} q^{\frac{7}{8}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= \mathrm{ch}^V_{(2,1)}(q^2)\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) + \mathrm{ch}^V_{(2,2)}(q^2)\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\ &+ \mathrm{ch}^V_{(3,2)}(q^2)\mathrm{ch}_{L(\Lambda_5)}(z;q^2) + \mathrm{ch}^V_{(4,2)}(q^2)\mathrm{ch}_{L(2\Lambda_6)}(z;q^2) \\ q^{\frac{7}{8}}\ \mathrm{ch}_{L(\Lambda_6)}(z;q) &= (\mathrm{ch}^V_{(1,1)}(q^2) + \mathrm{ch}^V_{(1,4)}(q^2))\mathrm{ch}_{L(\Lambda_7)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(1,2)}(q^2) + \mathrm{ch}^V_{(1,3)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_6)}(z;q^2) \end{aligned}$$

- $(E_6^{(1)})_2 \hookrightarrow (E_6^{(1)})_1$, $\widehat{c} = c(6) = \frac{6}{7}$

$$\begin{aligned} q^{\frac{3}{4}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= (\mathrm{ch}^V_{(2,1)}(q^2) + \mathrm{ch}^V_{(2,6)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(2,5)}(q^2))\mathrm{ch}_{L(\Lambda_6)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,3)}(q^2) + \mathrm{ch}^V_{(2,4)}(q^2))\mathrm{ch}_{L(\Lambda_1+\Lambda_5)}(z;q^2) \\ q^{\frac{3}{4}}\ \mathrm{ch}_{L(\Lambda_1)}(z;q) &= (\mathrm{ch}^V_{(2,1)}(q^2) + \mathrm{ch}^V_{(2,6)}(q^2))\mathrm{ch}_{L(2\Lambda_5)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(2,5)}(q^2))\mathrm{ch}_{L(\Lambda_4)}(z;q^2) \\ &+ (\mathrm{ch}^V_{(2,3)}(q^2) + \mathrm{ch}^V_{(2,4)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_1)}(z;q^2) \end{aligned}$$

- $(G_2^{(1)})_2 \hookrightarrow (G_2^{(1)})_1$, $\widehat{c} = c(9) = \frac{14}{15}$

$$\begin{aligned}
q^{\frac{7}{20}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= (\mathrm{ch}^V_{(1,2)}(q^2) + \mathrm{ch}^V_{(8,2)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(7,2)}(q^2))\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(3,2)}(q^2) + \mathrm{ch}^V_{(6,2)}(q^2))\mathrm{ch}_{L(2\Lambda_2)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,2)}(q^2) + \mathrm{ch}^V_{(5,2)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_2)}(z;q^2) \\
q^{\frac{7}{20}}\ \mathrm{ch}_{L(\Lambda_2)}(z;q) &= (\mathrm{ch}^V_{(1,4)}(q^2) + \mathrm{ch}^V_{(8,4)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,4)}(q^2) + \mathrm{ch}^V_{(7,4)}(q^2))\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(3,4)}(q^2) + \mathrm{ch}^V_{(6,4)}(q^2))\mathrm{ch}_{L(2\Lambda_2)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,4)}(q^2) + \mathrm{ch}^V_{(5,4)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_2)}(z;q^2)
\end{aligned}$$

- $(F_4^{(1)})_2 \hookrightarrow (F_4^{(1)})_1$, $\widehat{c} = c(10) = \frac{52}{55}$

$$\begin{aligned}
q^{\frac{13}{20}}\ \mathrm{ch}_{L(\Lambda_0)}(z;q) &= (\mathrm{ch}^V_{(2,1)}(q^2) + \mathrm{ch}^V_{(2,10)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,2)}(q^2) + \mathrm{ch}^V_{(2,9)}(q^2))\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,3)}(q^2) + \mathrm{ch}^V_{(2,8)}(q^2))\mathrm{ch}_{L(2\Lambda_4)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,4)}(q^2) + \mathrm{ch}^V_{(2,7)}(q^2))\mathrm{ch}_{L(\Lambda_3)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(2,5)}(q^2) + \mathrm{ch}^V_{(2,6)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_4)}(z;q^2) \\
q^{\frac{13}{20}}\ \mathrm{ch}_{L(\Lambda_4)}(z;q) &= (\mathrm{ch}^V_{(4,1)}(q^2) + \mathrm{ch}^V_{(4,10)}(q^2))\mathrm{ch}_{L(2\Lambda_0)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,2)}(q^2) + \mathrm{ch}^V_{(4,9)}(q^2))\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,3)}(q^2) + \mathrm{ch}^V_{(4,8)}(q^2))\mathrm{ch}_{L(2\Lambda_4)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,4)}(q^2) + \mathrm{ch}^V_{(4,7)}(q^2))\mathrm{ch}_{L(\Lambda_3)}(z;q^2) \\
&+ (\mathrm{ch}^V_{(4,5)}(q^2) + \mathrm{ch}^V_{(4,6)}(q^2))\mathrm{ch}_{L(\Lambda_0+\Lambda_4)}(z;q^2)
\end{aligned}$$

(Note: the simply-laced cases are also discussed in [2].)
Proof: The completeness of the list can be shown either through a case by case verification or by observing that the coset charge $\widehat{c}^{\mathfrak{g}}(k;j)$ is the same as that of the standard coset

$$\widehat{\mathfrak{g}}_{jk} \hookrightarrow \underbrace{\widehat{\mathfrak{g}}_k \oplus \ldots \oplus \widehat{\mathfrak{g}}_k}_{j}$$

(through the diagonal embedding) and then applying the results of [5]. Since, for given $\widehat{c}^{\mathfrak{g}}(k;j)$ and k there are only a finite number of unitary representations of $V_{\hat{c}} \oplus \widehat{\mathfrak{g}}_{jk}$, the branching rules are fixed up to a finite number of (integer) factors. These factors can be determined by examining the first few terms in the series expansions (in q) of the

characters. As a check on our calculations we verified, using the results of [6], that the asymptotics of the characters are consistent with the obtained branching rules.□

Note that, even though the coset central charges are the same as those of the standard coset construction applied to the diagonal cosets $\widehat{\mathfrak{g}}_2 \hookrightarrow \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_1$, the branching rules are quite different. While for the diagonal cosets only those V_c-modules occur that extend to modules of the $\mathcal{W}[\mathfrak{g}]$-algebra [7,8], in the present case typically the complement of those in the Kac table occur. We expect that those modules are modules of an extension of the Virasoro algebra along the lines of [8] as well.

One important application concerns the computation of the root multiplicities of hyperbolic Kac-Moody algebras or, more generally, generalized Kac-Moody algebras [1,9,10]. As an example, consider the above branching rule for $\mathfrak{g} = E_8$. We find

$$\mathrm{ch}^V_{(2,2)}(q^2)\,\mathrm{ch}_{L(\Lambda_1)}(z;q^2) \;=\; \left(q\,\mathrm{ch}_{L(\Lambda_0)}(z;q)\right)\Big|_{\mathrm{even}}\,, \tag{17}$$

where the subscript ‘even’ denotes that only the even powers of q in the series expansion on the right hand side of the equation contribute. Thus, in particular,

$$\begin{aligned} q\,b^{\Lambda_1}_{2\Lambda_0}(q^2) \;&=\; q\,b^{\Lambda_1}_{\Lambda_7}(q^2) \;=\; \frac{\phi(q^2)}{\phi(q^4)}\,b^{\Lambda_0}_{\Lambda_0}(q)\Big|_{\mathrm{odd}}\,,\\ b^{\Lambda_1}_{\Lambda_1}(q^2) \;&=\; \frac{\phi(q^2)}{\phi(q^4)}\,b^{\Lambda_0}_{\Lambda_0}(q)\Big|_{\mathrm{even}}\,, \end{aligned} \tag{18}$$

where $b^\Lambda_\lambda(q)$ denote the Kac-Peterson string functions [1] (normalized such that $b^\Lambda_\lambda(q) = \mathcal{O}(1)$), and where we have used

$$\mathrm{ch}^V_{(2,2)}(q) \;=\; q^{\frac{1}{16}}\,\frac{\phi(q^2)}{\phi(q)}\,, \qquad\qquad \phi(q) \;=\; \prod_{n\geq 1}(1-q^n)\,. \tag{19}$$

Or, equivalently,

$$b^{\Lambda_1}_{\Lambda_1}(q^2) + q\,b^{\Lambda_1}_{\Lambda_7}(q^2) \;=\; \frac{\phi(q^2)}{\phi(q^4)}\,b^{\Lambda_0}_{\Lambda_0}(q) \;=\; \frac{\phi(q^2)}{\phi(q^4)\phi(q)^8}\,, \tag{20}$$

which is one of the crucial identities in the computation of the level-2 root multiplicities of the hyperbolic Kac-Moody algebra E_{10} in [9]. While in [9] equation (20) required a lengthy derivation using the modular properties of the characters, we see that here it follows quite easily from a coset construction. It is conceivable that a proper generalization of this coset construction along the lines of [8] will be useful in the computation of higher level root multiplicities.

A second application is the construction of modular invariant sesquilinear combinations of either $\widehat{\mathfrak{g}}_{jk}$ or V_c characters from known modular invariants of $\widehat{\mathfrak{g}}_k$ along the lines of, e.g., [11,12,6]. Actually, due to the nature of (10), these will be invariants of the principal

congruence subgroup $\Gamma(j)$ rather than of the full $PSL(2,\mathbb{Z})$ (and of $\Gamma(2)$ for the examples of Theorem 4). We refrain from giving a complete list – the interested reader can easily work out examples.

Another potential application, which motivated the present work, is related to the work of Nakajima [13]. It was suggested in [14] that the fact that the cohomology of the moduli space of $U(k)$ instantons on the ALE space of type A_N carries an $(A_N^{(1)})_k$-module structure [13] can be understood in the context of heterotic string duality by restricting the $\widehat{\mathfrak{g}}_1$-action on a free field Fock space to the oscillators mod k. The above results give a precise description of the decomposition under the embedding $\widehat{\mathfrak{g}}_k \hookrightarrow \widehat{\mathfrak{g}}_1$.

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